1 (Optimal Mass Transport). Suppose $\mathbb{D}$ is the unit disk in the plane, $P = \{p_1, p_2, \cdots, p_n\}$ is a discrete planar point set. Each point p_i is associated with a weight r_i, the power distance between any point $p \in \mathbb{R}^2$ to p_i is defined as

$$Pow(p, p_i) = |p - p_i|^2 + r_i.$$

The power Voronoi diagram is a partition of the whole plane

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} W_i, \ W_i = \{p \in \mathbb{R}^2 | Pow(p, p_i) \leq Pow(p, p_j), \forall 1 \leq j \leq n\}.$$

The power Vornoi diagram induces a cell decomposition of $\mathbb{D}$,

$$\mathbb{D} = \bigcup_{i=1}^{n} W_i \cap \mathbb{D},$$

suppose the area of each cell $\mathbb{D} \cap W_i$ is A_i. Construct a mapping $\varphi : \mathbb{D} \to P$, such that each cell $W_i \cap \mathbb{D}$ is mapped to the point p_i,

$$\varphi : W_i \cap \mathbb{D} \mapsto p_i, \ \forall 1 \leq i \leq n.$$

(1) Suppose given another cell decomposition

$$\mathbb{D} = \bigcup_{i=1}^{n} \tilde{W}_i \cap \mathbb{D},$$

and construct a mapping $\tilde{\varphi}$, such that

$$\tilde{\varphi} : \tilde{W}_i \cap \mathbb{D} \mapsto p_i,$$

and the area of each cell $\tilde{W}_i \cap \mathbb{D}$ equals to A_i as well. The L^2 transportation cost of φ is defined as

$$E(\varphi) := \int_{\mathbb{D}} |p - \varphi(p)|^2 dA,$$

show that the mapping φ is optimal, i.e.

$$E(\varphi) \leq E(\tilde{\varphi}).$$

(2) Show that there exists real numbers $h_1, h_2, \cdots, h_n$, which determine n planes

$$\pi_i(p) := \langle p, p_i \rangle + h_i,$$

the upper envelope of the planes $\{\pi_i\}$ is the graph of the convex PL function

$$f(p) = \max_{1 \leq i \leq n} \pi_i(p).$$

The power Voronoi diagram is induced by the projection of the upper envelope of these planes $\{\pi_i, i = 1, 2, \cdots, n\}$